according to the symmetric group of order 6. Hence each of the required groups contains invariantly the direct product of the group of order 7 and a group of order 4. If such a group has either one or four subgroups of order 3 it must be the direct product of the group of order 7 and a group of order 24 which has three subgroups of order 8 and either one or four subgroups of order 3. Hence there are just eight such groups, seven of them involve only one subgroup of order 3, while the remaining one involves four such subgroups.

There cannot be either 7 or 28 subgroups of order 3, since such a group would include 56 operators that are commutative with an operator of order 7 and therefore it could not contain just three conjugate subgroups of order 56 as was stated above.

3. Groups Containing Seven Subgroups of Order Eight.

The subgroups of order 8 are permuted according to the transitive group of degree 7 and order 14 when the number of subgroups of order 3 is either one or four. When the number of these subgroups is either 7 or 28 the subgroups of order 8 are permuted according to the group of order 42 and degree 7. Each of the groups under consideration contains an invariant subgroup of order 4. In the former case there must be an invariant subgroup of order 84, which is the direct product of the group of order 7 and one of the three groups of order 12 which include only one group of order 4.

Three of the required groups may be obtained by forming the direct products of the group of order 14 and degree 7 multiplied by one of the three groups of order 12 which have just been mentioned. When the given invariant subgroup of order 84 is cyclic there are three additional groups.* In one of these the subgroups of order 8 are cyclic, in the second they are the quaternion group, and in the third each of them contains just two operators of order 4. Hence, there are just four groups of order 168 that contain 7 subgroups of order 8 and also an operator of order 84. When the invariant subgroup of order 84 is noncyclic and abelian there are two additional groups, whose subgroups of order 8 are the abelian group which has two invariants and the group which can be represented on four letters, respectively. Finally, when the given subgroup of order 84 is nonabelian there is no group in addition to the direct product mentioned above. The total number of groups of order 168, which contain an invariant subgroup of order 12 and just seven subgroups of order 8 is therefore 8.

By forming the direct products of the metacyclic group of degree 7 and the groups of order 4, we obtain two groups containing seven subgroups of order three. Each of the groups which contain seven such subgroups includes the direct product of the semimetacyclic group of degree 7 and one of the groups of order 4. When this group of order 4 is cyclic there are three additional groups, whose subgroups of order 8 are distinct. When this group of order 4 is noncyclic there are two more groups. Hence there are just 7 groups containing 7 subgroups of order 3.

Transactions of the American Mathematical Society, Vol. 2, 1901, page 264.

When there are 28 subgroups of order 3, they are permuted according to a simply isomorphic group of order 168, which contains a subgroup of order 84. As the operators of order 2 in this subgroup cannot transform a subgroup of order 3 into itself the required groups (or group) must include operators of order 2 which are not in the subgroup of order 84. Hence there is only one such group, and the total number of groups of order 168 which contain seven subgroups of order 8 is 16.

4. Groups Containing Twenty-One Subgroups of Order Eight.

Such a group contains three conjugate subgroups of order 56, which it permutes according to the symmetric group of order 6. Each of these subgroups contains 7 groups of order 8, which it permutes according to a group of order 14. Hence it contains just one invariant subgroup of order 4, while the entire group G contains only one invariant subgroup of order 28. Since the quotient group of G with respect to this invariant subgroup is the symmetric group of degree 3, G cannot contain more than four subgroups of order 3; for, if there were more than four such subgroups, the operators of order 7 would be permuted according to the cyclic group of order 6 and hence this group would be a quotient group.

The invariant subgroup of order 28 includes either seven or just one subgroup of order 4. In the former case they are permuted under G according to the group of order 14, as there cannot be an invariant subgroup of order 4 in this case, and hence there must be an invariant subgroup of order 12. Since there is also such an invariant subgroup in the latter case, each of the groups under consideration contains an invariant subgroup of order 12, and hence also the direct product of the group of order 7 and some group of order 12.

When this direct product is cyclic, there are four such groups, which may be distinguished by their subgroups of order 8. When the direct product is the other abelian group of order 84, there are three additional groups. In two of these the subgroups of order 8 are abelian while those of the third can be represented on four letters. Each of these seven groups contains only one abelian group of order 84. It remains to determine the groups which do not include an abelian subgroup of order 84.

When the subgroup of order 12 is the alternating group there is only one G. Hence there is only one group of order 168 that includes 21 subgroups of order 8 and 4 subgroups of order 3. As each of the remaining groups contains three conjugate subgroups of order 4, its invariant subgroup of order 28 includes seven subgroups of order 4. When the subgroup of order 12 includes six operators of order 4, there are four additional groups; two of them contain just four operators of order 4 in each subgroup of order 8, while the non-abelian subgroups of order 8 in the other two are distinct. The first two of these four groups may be distinguished by the fact that only the first of them contains operators of order 12.

The only case which remains is when the subgroup of order 12 includes 7 operators of order 2. In this case there are again four groups. Two of these contain distinct abelian subgroups of order 8, while the other two include the

group of order 8 which can be represented on four letters. The latter may again be distinguished by the fact that only one of them contains operators of order 12. This concludes the examination of all the possible cases and proves that the total number of groups of order 168 that have 21 subgroups of order 8 is the same as the number of those containing 7 such subgroups, viz. 16. *The total number of groups of order* 168 *is therefore* 57. Only one of these is insolvable.

ON LIMITS.

By DR. ARNOLD EMCH, University of Colorado.

In most of the elementary text-books of algebra and calculus the chapters on limits are treated in a rather indefinite manner. I suspect that the reason for this deficiency lies partly in the semi-philosophical nature of the subject, partly in the neglect of the authors to apply the results of the theory of functions to limitting processes.

As an example I mention the frequently occurring definition of the *limit of a variable.*

(*a*) "When according to its law of change, a variable approaches indefinitely near a constant, but can never reach it, the constant is called the limit of the variable."

This definition restricts the variable to the members of a sequence as it will appear from the following proposition given by Harkness and Morley :*

I. "The numbers $\xi_1, \xi_2, \xi_3, \ldots$ of a sequence are said to tend to the limit a, when to every positive number ε there corresponds a positive integer μ such that for ξ_μ and for all later members ξ_n of the sequence we have

$$|\xi_n - a| < \varepsilon.$$"

a itself does not belong to the sequence. Terms in the sequence of all positivs integers 1, 2, 3, ...—, ∞ does not belong to the sequence, although ∞ is the limit of the sequence. Similarly, in the sequence

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

0, which is its limit, does not belong to the sequence. Notice also that 0 and ∞, which shall be defined presently, are among the values which a limit can have.

An infinitely large quantity, or the symbol ∞ is defined as the indefinite quantity A which satisfies the inequality

**Introduction to Analytic Functions*, page 67.
See also Burkhardt: *Functionentheorie*, Vol. I., page 68.

$$A > N$$

for all positive numbers N no matter how large. Here, N may be either an integer or any other positive real number. $-\infty$ may be defined in a similar manner.

For a sharp conception of infinite limits as used in the theory of functions and in geometry, a distinction between *absolute infinity* and *variable infinity* (Werdendes Unennlich) should be made.*

Designating the first infinity by ∞, the second by (∞) we have the definitions

$$\frac{1}{\infty} = 0 \text{ (absolute zero)},$$

$$\frac{1}{(\infty)} = \varepsilon \text{ (infinitesimal)}.$$

An infinitesimal ε itself is defined by the inequality

$$|\varepsilon| < |a|,$$

for all values of a no matter how small. Conversely

$$\frac{1}{\varepsilon} = (\infty); \quad \frac{1}{0} = \infty.\dagger$$

Take now a variable y whose values depend upon an independent variable x of which we assume that it may assume all real values of the interval between a and b. This interval may be closed, *i e.*, a and b belong to the interval; it may be open, in which case a and b are excluded; it may be closed at a and open at b, or conversely. In signs we have the four cases

$$a \leqq x \leqq b,$$
$$a < x < b,$$
$$a \leqq x < b,$$
$$a < x \leqq b.$$

Among the values which a and b may have we also include the symbols 0, $+\infty$, $-\infty$, and as a matter of course ε, $+(\infty)$, $-(\infty)$. From the physical standpoint 0, $+\infty$, $-\infty$ are pure abstractions of the human mind, like the hypotheses of the infinite divisibility of space. A real variable x which successively may take all values of an interval is said to be continuous. For the subsequent applications I add the following propositions:

*Encyklopaedie der Mathematischen Wissenschaften, Vol. I., Heft 1., IA3., page 68.
†Burkhardt, loc. cit., pages 27—29.

II. A function $f(x)$ is called *continuous* at a certain place x_0, if the difference

$$f(x_0+h)-f(x_0),$$

which is a function of h, becomes infinitely small with h. (If x is not variable unrestrictedly, the variability of h has to be restricted to such values for which x_0+h belongs to the values of x.

III. A function is called *continuous within an interval* if it is continuous at every place of the interval.

IV. A function which is continuous within an interval reaches its upper and lower limits within this interval.*

Thus taking $y=1/x$, $0<x\leqq\infty$, (∞) is an indefinite upper limit, while 0 is the lower limit. For $-\infty\leqq x<0$, the upper and lower limits are 0 and $-(\infty)$ respectively. The point 0 is a point of discontinuity, for in this case $f(0+h)-f(0)$, as h approaches 0, may assume any value. However,

$$\lim_{h=0}\{f(\varepsilon+h)-f(\varepsilon)\}=0,$$

i. e. $y=1/x$ is continuous for infinitesimal values of x. The difference between this example and the sequence $\frac{1}{1}, \frac{1}{2}\cdot\frac{1}{3}, \frac{1}{4}, \ldots$ is now apparent. According as a variable is continuous or discontinuous within an interval it may or may not reach its limit, or a limit. It may even be alternately greater or less than its limit, without reaching it, as in the case of

$$\sum_0^n(-1)^n a_n \ (n \text{ integer}, \ a_n\text{'s decreasing}),$$

or the convergents of a continued fraction.

Hilbert in a lecture on "Mathematical Problems" before the International Mathematical Congress at Paris, 1900, said :†

"In der Geometrie gelingt der Nachweis der Widerspruchslosigkeit der Axiome dadurch, dass man einen geeigneten Bereich von Zahlen konstruiert, derart, dass den geometrischen Axiomen analoge Beziehungen zwischen den Zahlen dieses Bereiches entsprechen und dass demnach jeder Widerspruch in den Folgerungen aus geometrischen Axiomen auch in der Arithmetik jenes Zahlenbereiches erkennbar sein müsste."

With this purpose in view I shall apply the theorems on limits to a few geometrical problems.

The previous statements concerning the limits of $y=1/x$ may also be expressed thus :

$$\lim_{x=\pm\varepsilon}\left(\frac{1}{x}\right)=\pm(\infty); \quad \lim_{x=\pm\infty}\left(\frac{1}{x}\right)=0.$$

*Burkhardt, loc. cit., page 71, and Harkness and Morley, page 74.
†*Archiv der Mathematik und Physik*, III Series, Vol. I., page 55.

Geometrically $y=1/x$ represents an hyperbola having the coördinate-axes as asymptotes. The variable x can evidently assume all values of the closed interval $-\infty \leqq x \leqq +\infty$. As x approaches 0 indefinitely (infinitesimally, ε) from positive values, y reaches a positive infinite value, or becomes (∞), infinitely large. If x approaches 0 indefinitely from negative values, y reaches $-(\infty)$.

Now the behavior of y at $x=\pm\infty$, $(\pm(\infty))$, is exactly the same as that of x at $y=\pm\infty$, $(\pm(\infty))$. Thus, $y=0$ for $x=\pm\infty$ gives us the notion of projective geometry that the asymptotes of an hyperbola are the tangents at its infinitely distant points. Arithmetically, an hyperbola has four infinite points, ε, (∞), ε, $-(\infty)$, (∞), ε, $-(\infty)$, ε. This however does not agree with the conceptions of an asymptote as a tangent and it is therefore necessary to assume the axiom that *a straight line has only one infinite point* and that *a plane has only one infinite straight line. This axiom is equivalent with the axiom of absolute infinity.* Indeed, $y=\frac{1}{0}=\infty$ and $y=1/\infty=0$ now signifies that the hyperbola has only two infinite points and that the asymptotes are the tangents at these two points.

We have now perfect agreement with the results of perspective collineation. If a circle intersects the axis of collineation s at two points A and B, its perspective is an hyperbola whose asymptotes are the transformed tangents at A and B. Perspective is a *one to one* correspondence, so that from this point of view the asymptotes necessarily have to be considered as limitting positions of tangents. By an inversion, which realizes also a (1, 1) correspondence, the hyperbola $xy=1$ is transformed into the lemniscate

$$xy=(x^2+y^2)^2.$$

This curve is continuous at $x=y=0$; to the asymptotes correspond the tangents to the lemniscate at the origin. Thus, in the theory of functions and in geometry the treatment of functions at $x=y=0$, respectively $z=0$, is not essentially different from that at infinity. In geometry a collineation reduces a problem concerning infinities to one of finite regions. In the theory of functions this is done by the inversion $u=1/z$. In the theory of assemblages,* ∞ is a place of accumulation (Hänfungstelle) for the system 1, 2, 3, 4, ... If we take a *non-enumerable* system consisting of all real numbers, ∞ is not more of a place of accumulation than any other point. As a second example take a polygon of n sides inscribed to a circle of unit-diameter. For the perimeter of this polygon we have

$$S_n=n\sin\frac{\pi}{n}.$$

For $n=2, 3, \ldots$ we have a sequence whose lower limit is 2 and whose upper limit is

*See E. Borel: *Lecons sur la theorie de fonctions*, Vol. 1., pages 1—49.

$$\lim_{n=\infty}(S_n)=(\infty.0)=\lim_{n=\infty}\left\{\frac{-\frac{\pi}{n^2}}{-\frac{1}{n^2}}\right\}=\pi.$$

This limit does not belong to the sequence, *i. e.* none of the polygons is equal to the circumference. We may, however, say correctly, as Sophus Lie does,* that in the limit-process (Grenzübergang) the polygon goes over into the circle.

The matter stands quite different if we consider the function

$$y=x\sin\frac{\pi}{x}$$

of the real continuous variable x. The function is continuous within the interval

$$0<x\leqq\infty,$$

and defined within the interval

$$0\leqq x<\infty.$$

It is noticed that $x=0$ is a singular point for the function, although $y=0$ for $x=0$. We have

$$\frac{dy}{dx}=\sin\frac{\pi}{x}-\frac{\pi}{x}\cos\frac{\pi}{x}.$$

As x approaches 0 indefinitely, $\frac{dy}{dx}$ changes infinitely many times abruptly from $+(\infty)$ to $-(\infty)$. As x passes through zero, y reaches the limit 0, owing to the circumstance that x appears as a factor in the expression for y. Geometrically, we may say that the curve represented by $y=x\sin\frac{\pi}{x}$ has an infinite number of infinitely small oscillations infinitely close to the origin.

For $x=\infty$, y is not defined, but its value is $\lim_{x=\infty}(y)=\lim_{x=\infty}\left(x\sin\frac{\pi}{x}\right)=\pi$. This upper limit† of y is attained for $x=\infty$. The straight line $y=\pi$ is an asymptote of the given curve and may be considered as a tangent at its infinite point. The inverse curve (origin as center) is continuous $x=0$; discontinuous at $x=\infty$.

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*Beruehrungstransformationen, Vol. I., page —.

†Within the interval $(2\leqq x\leqq\infty)$.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

150. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A commission merchant sold $\$W$,=\$4750 worth of wheat. After deducting his commission at $m\%$,=3%, purchased with the proceeds a draft at d,=60 days at $r\%$,=10%, interest, and at $p\%$,=$\frac{3}{4}$% premium. What was the face of the draft?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.

$\$W(100\% - m\%) = \$4750 \times .97 = \$4607.50.$

$(d+3)$ days=63 days, $\frac{(d+3)r\%}{36} = \frac{63 \times 10\%}{36} = .0175.$

$100\% + p\% - \frac{(d+3)r\%}{36} = 1.0075 - .0175 = .99.$

$\$6407.50 \times .99 = \$6343.425.$

151. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

A merchant marked a lot of goods $m\%$=20%, above cost; but, in consequence of a rise in the market price, he marked up the goods $n\%$=10%, on the marked price. What per cent. was the last selling price of the goods? What would be his gain on sales amounting to $\$S$=\$5780.50?

Solution by JAMES F. LAWRENCE, A. B., Instructor in Mathematics, Rogers Academy, Rogers, Ark.

1. Let $r\%$=100%=cost.
2. $\therefore$ $r\% + m\%$=marked price.
3. Let $p\%$=100%=marked price=$r\% + m\%$.
4. $1\% = \frac{r+m}{p}\%$.
5. $(p+n)\% = \frac{(r+m)(p+n)}{p}\%$=132%=selling price.
6. $\therefore$ 32%=gain on cost price.
7. But $\frac{(r+m)(p+n)}{p}\%$=$\$S$=\$5780½.
8. $\therefore$ r=100% $= \frac{rpS}{(r+m)(p+n)}$=\$4379⅛=cost.
9. $\therefore$ \$1401½=\$5780½−\$4379⅛=gain.

Also solved by *T. T. DAVIS*, Principal of High School, Portland Oregon; *G. B. M. ZERR*, *C. A. LINDEMAN*, and *P. S. BERG*.

ALGEBRA.

129. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, O.

$$\begin{vmatrix} 1-\binom{m-1}{0} & -\binom{m-1}{1} & \cdots & \binom{m-1}{m-2} \\ 1 & 1-\binom{m-2}{0} & \cdots & \binom{m-2}{m-3} \\ 1 & 0 & 1 & \cdots \binom{m-3}{m-4} \\ \cdots\cdots & \cdots & \cdots\cdots & \\ 1 & 0 & 0 & 1 \quad \binom{1}{0} \\ 1 & 0 & 0 \ldots . 0 & 1 \end{vmatrix} = e = \sum_{n=\infty}^{n=1} \frac{n^m}{n!}, \text{ where } \binom{m-2}{k} = \frac{(n-2)\ldots(m-k-1)}{k!}$$

Solution by the PROPOSER.

Let $f(x)=e^{e^x}=1+e^x+\frac{e^{2x}}{2!}+\frac{e^{3x}}{3!}+\frac{e^{4x}}{4!}+\frac{e^{5x}}{5!}+\ldots+\frac{e^{rx}}{r!}+\ldots.$

$$=1+(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots.)$$

$$+\frac{1}{2!}(1+2x+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\frac{(2x)^4}{4!}+\frac{(2x)^5}{5!}+\ldots.)$$

$$+\frac{1}{3!}(1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\ldots)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{1}{r!}(1+rx+\frac{(rx)^2}{2!}+\frac{(rx)^3}{3!}+\ldots.)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$=e+\sum_{r=1}^{r=\infty} a_r x^r, \text{ where } a_r=\frac{1}{r!}\sum_{k=1}^{k=\infty}\frac{k^r}{k!}.$$

But $f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots.$

$\therefore f^m(0)=\sum_{k=1}^{k=\infty}\frac{k^m}{k!}\ldots.(1).$

Let $y=e^{e^x}$, then $\log y=e^x$.

$\therefore \frac{dy}{dx} = ye^x$. And by Leibnitz's Formula,

$$f^m(0) = f^{m-1}(0) + (m-1)f^{m-2}(0) + \frac{(m-1)(m-2)}{2!} f^{m-3}(0) + \ldots$$

$$+ (m-1)f'(0) + f(0) = \sum_{k=1}^{k=\infty} \frac{k^m}{k!}. \text{ [by (1)]} \quad \ldots(2).$$

Or, letting x_m represent $f^m(0)$, etc.,

$$x_m - x_{m-1} - C_{m-1,1}x_{m-2} - \ldots - C_{m-1,1}x_1 = f(0) = e.$$

$$x_{m-1} - x_{m-2} - C_{m-2,1}x_{m-3} \ldots - C_{m-2,1}x_1 = e$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_2 - x_1 = e$$

$$x_1 = e$$

$$\therefore x_m = \begin{vmatrix} e & -\binom{m-1}{0} & -\binom{m-1}{1} & \ldots\ldots\ldots & -\binom{m-1}{m-2} \\ e & 1 & -\binom{m-2}{0} & \ldots\ldots\ldots & -\binom{m-2}{m-3} \\ e & 0 & 1 & \ldots\ldots\ldots & -\binom{m-3}{m-4} \\ e & 0 & 0 & 1 \ldots\ldots & -\binom{m-4}{m-5} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ e & 0 & 0 & 0 \ldots 1 & -\binom{1}{0} \\ e & 0 & 0 & 0 \ldots\ldots 0 & 1 \end{vmatrix}$$

$$\therefore \begin{vmatrix} 1 & -\binom{m-1}{0} & -\binom{m-1}{1} & \ldots\ldots\ldots & -\binom{m-1}{m-2} \\ 1 & 1 & -\binom{m-2}{0} & \ldots\ldots\ldots & -\binom{m-2}{m-3} \\ 1 & 0 & 1 & \ldots\ldots\ldots & -\binom{m-3}{m-4} \\ 1 & 0 & 0 & 1 \ldots\ldots & -\binom{m-4}{m-5} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 1 & 0 & 0 & 0 \ldots\ldots 1 & -\binom{1}{0} \\ 1 & 0 & 0 & 0 \ldots\ldots 0 & 1 \end{vmatrix} e = \sum_{k=1}^{k=\infty} \frac{k^m}{k!} \ldots\ldots (3).$$

By using (2) or (3) the following special relations have been obtained :

$$2e=\sum_{k=1}^{k=\infty}\frac{k^2}{k!},\quad 5e=\sum_{k=1}^{k=\infty}\frac{k^3}{k!},\quad 15e=\sum_{k=1}^{k=\infty}\frac{k^4}{k!},\quad 52e=\sum_{k=1}^{k=\infty}\frac{k^5}{k!},\quad 203e=\sum_{k=1}^{k=\infty}\frac{k^6}{k!},\quad 877e=\sum_{k=1}^{k=\infty}\frac{k^7}{k!},$$

$$4140e=\sum_{k=1}^{k=\infty}\frac{k^8}{k!},\quad 21147e=\sum_{k=1}^{k=\infty}\frac{k^9}{k!},\quad 115975e=\sum_{k=1}^{k=\infty}\frac{k^{10}}{k!}.$$

Also solved by *G. B. M. ZERR.*

GEOMETRY.

160. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

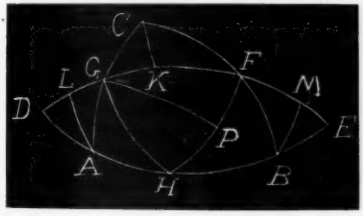
Let GFH be the spherical triangle formed by joining the mid-points of the sides of the spherical triangle ABC; E the spherical excess of ABC; β, p the base and altitude of GFH. Prove $\sin\frac{1}{2}E=\sin\beta\sin p$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let $BA=c$, $GF=\gamma$, $FH=\beta$, $GH=\delta$, $GP=p$.

Draw AL, BM, CK perpendicular to $DGFC$. Now $DABE=DGFE=\pi$.

$AD=BE=\frac{1}{2}(\pi-c)$, $DL=ME$, $LG=CK$, $KF=FM$.



$\therefore 2(DL+GK+KF)=\pi$. Also $2GK+2KF=2\gamma$.

$\therefore 2DL+2\gamma=\pi$, or $DL=\frac{1}{2}\pi-\gamma$.

$\angle DAL=\angle EBM$, $\angle LAG=\angle GCK$, $\angle KCF=\angle FBM$.

$\therefore 2\angle DAL+C+A+B=2\pi$.

$\therefore \angle DAL=\pi-\frac{1}{2}(A+B+C)=\pi-s=\frac{1}{2}(\pi-E)$.

$\cos DAL=\sin D\cos DL$. $\cos\frac{1}{2}(\pi-E)=\sin D\cos(\frac{1}{2}\pi-\gamma)$.

$\therefore \sin\frac{1}{2}E=\sin D\sin\gamma$.

Now $\sin D:\sin DFH=\sin\beta:\sin DH$. But $DH=\frac{1}{2}\pi$.

$\therefore \sin D=\sin\beta\sin DFH$. But $\sin p=\sin\gamma\sin DFH$.

$\therefore \sin D=(\sin\beta\sin p)/\sin\gamma$.

$\therefore \sin\frac{1}{2}E=\sin\beta\sin p$.

Also solved by *J. SCHEFFER* and *L. C. WALKER.*

161. Proposed by MARCUS BAKER, U. S. Coast and Geodetic Survey Office, Washington, D. C.

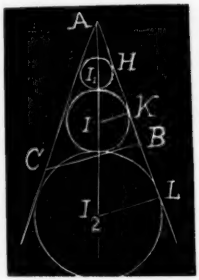
A circle, radius r, is inscribed in a triangle ABC. In the angles A, B, and C are inscribed circles each touching two sides and the inscribed circle. There are six such circles. The first group of three have their centers between the in-centers and the vertices, and the second group of three does not. Let r_a, r_b, r_c denote the radii of the first group. Then this well known relation holds: $r=\sqrt{(r_ar_b)}+\sqrt{(r_br_c)}+\sqrt{(r_cr_a)}$. Let R_a, R_b, R_c denote the radii of the second group. Then this relation holds:

$$\frac{1}{r}=\frac{1}{\sqrt{(R_aR_b)}}+\frac{1}{\sqrt{(R_bR_c)}}+\frac{1}{\sqrt{(R_cR_a)}}.$$

Required proof.

I. Solution by LON C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Let ABC be a triangle, I the center of the in-circle with radius r. The circles with radii r_a, R_a will have centers I_1, I_2, and points of contact with the in-circle on the straight line AI. Draw the radii I_1H, IK, I_2L. The length of IA is



$$r+r_a+r_a\operatorname{cosec}\tfrac{1}{2}A=r\operatorname{cosec}\tfrac{1}{2}A\ ;\ \text{thence}$$

$$r_a=\frac{r(1-\sin\frac{1}{2}A)}{1+\sin\frac{1}{2}A}=\frac{r(\cos\frac{1}{4}A-\sin\frac{1}{4}A)^2}{(\cos\frac{1}{4}A+\sin\frac{1}{4}A)^2},$$

$$\text{and } \sqrt{(r_ar_b)}=\frac{r(\cos\frac{1}{4}A-\sin\frac{1}{4}A)\,(\cos\frac{1}{4}B-\sin\frac{1}{4}B)}{(\cos\frac{1}{4}A+\sin\frac{1}{4}A)(\cos\frac{1}{4}B+\sin\frac{1}{4}B)}$$

$$=\frac{r\cos\dfrac{A+\pi}{4}\cos\dfrac{B+\pi}{4}\cos\dfrac{C+\pi}{4}}{\cos\dfrac{A-\pi}{4}\cos\dfrac{B-\pi}{4}\cos\dfrac{C-\pi}{4}}=\frac{r(\cos\frac{1}{2}A+\cos\frac{1}{2}B-\cos\frac{1}{2}C)}{\cos\frac{1}{2}A+\cos\frac{1}{2}B+\cos\frac{1}{2}C}.$$

In like manner can be found $\sqrt{(r_br_c)}$ and $\sqrt{(r_cr_a)}$. Then by adding,

$$r=\sqrt{(r_ar_b)}+\sqrt{(r_br_c)}+\sqrt{(r_cr_a)}.$$

The length of I_2A is $R_a+r+\operatorname{cosec}\frac{1}{2}A=R_a\operatorname{cosec}\frac{1}{2}A$; thence

$$\frac{1}{R_a}=\frac{1-\sin\frac{1}{2}A}{r(1+\sin\frac{1}{2}A)},\ \text{as before,}$$

$$\frac{1}{\sqrt{(R_aR_b)}}=\frac{\cos\frac{1}{2}A+\cos\frac{1}{2}B-\cos\frac{1}{2}C}{r(\cos\frac{1}{2}A+\cos\frac{1}{2}B+\cos\frac{1}{2}C)}.$$

Similarly, can be found $\dfrac{1}{\sqrt{(R_bR_c)}}$ and $\dfrac{1}{\sqrt{(R_cR_a)}}$. Then by adding,

$$\frac{1}{r}=\frac{1}{\sqrt{(R_aR_b)}}+\frac{1}{\sqrt{(B_bR_c)}}+\frac{1}{\sqrt{(R_cR_a)}}.$$

II. Solution by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

It is clear that $r\operatorname{cosec}\frac{1}{2}A+r+R_a=R_a\operatorname{cosec}\frac{1}{2}A$.

$$\therefore \Sigma \frac{1}{\sqrt{(R_a R_b)}} = \frac{1}{r} \Sigma \sqrt{\frac{(1-\sin\tfrac{1}{2}A)(1-\sin\tfrac{1}{2}B)}{(1+\sin\tfrac{1}{2}A)(1+\sin\tfrac{1}{2}B)}}$$

$$= \frac{1}{r} \frac{\Sigma(\cos\tfrac{1}{2}A-\sin\tfrac{1}{2}A)(\cos\tfrac{1}{2}B-\sin\tfrac{1}{2}B)(\cos\tfrac{1}{2}C+\sin\tfrac{1}{2}C)}{\pi(\cos\tfrac{1}{2}A+\sin\tfrac{1}{2}A)(\cos\tfrac{1}{2}B+\sin\tfrac{1}{2}B)(\cos\tfrac{1}{2}C+\sin\tfrac{1}{2}C)}$$

$$= \frac{1}{r} \frac{\Sigma\cos\frac{A+\pi}{4}\cos\frac{B+\pi}{4}\cos\frac{C-\pi}{4}}{\pi\cos\frac{A-\pi}{4}}$$

$$= \frac{1}{r} \frac{\Sigma(\cos\frac{1}{2}A+\cos\frac{1}{2}B-\cos\frac{1}{2}C)}{\Sigma\cos\frac{1}{2}A} = \frac{1}{r}.$$

Also solved by *H. C. WHITAKER*, and *G. B. M. ZERR*.

162. Proposed by J. D. PALMER, Providence, Ky.

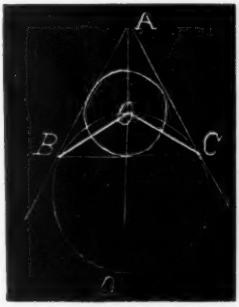
Given the distances from the vertices of a triangle, ABC, to the center of the in-circle, to construct the triangle.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

AO, BO, $CO=a$, b, c, respectively, where O is the center of the in-circle; BC, AC, $AB=x$, y, z, respectively. Let O_1 be the center of the ex-circle opposite A. Then

$$AO^2 = \frac{(p-x)^2}{\cos^2\frac{1}{2}A}, \text{ where } p=\tfrac{1}{2}(x+y+z).$$

$$\therefore AO^2 = \left[\frac{p-x}{p}\right]yz = yz - \frac{xyz}{p} = yz - 4Rr = a^2.$$



Similarly, $BO^2 = \left[\frac{p-y}{p}\right]xz = xz - 4Rr = b^2$.

$$CO^2 = \left[\frac{p-z}{p}\right]xy = xy - 4Rr = c^2.$$

$$\therefore AO.BO.CO = \frac{\triangle xyz}{p^2} = 4R^2r^2 \text{ or } 2\sqrt{(abc)} = 4Rr$$

$$\therefore a^2+2\sqrt{(abc)} = yz,\ b^2+2\sqrt{(abc)} = xz,$$

$$c^2 = 2\sqrt{(abc)} = xy.$$

$$\therefore x[a^2+2\sqrt{(abc)}] = y[b^2+2\sqrt{(abc)}] = z[c^2+2\sqrt{(abc)}]$$

$$=\sqrt{\{[a^2+2\sqrt{(abc)}][b^2+2\sqrt{(abc)}][c^2+2\sqrt{(abc)}]\}}.$$

This gives us the values of the sides.

Otherwise draw AO and produce AO to O_1 so that $OO_1=2\sqrt{(bc/a)}$. Upon OO_1 as diameter describe a circle. With O as a center and b as a radius describe an arc cutting the circle in B. Similarly, with O as center and c as radius, draw an arc cutting the circle in C. Join BC, AC, AB, then ABC is the triangle required. For O_1 is the ex-center opposite A by construction as follows:

$AO_1=p/\cos\frac{1}{2}A$. $\therefore$ $AOAO_1=yz=AO^2+2\sqrt{(AO.BO.CO)}$.

$\therefore$ $AO_1=AO+2\sqrt{[(BO.CO)/AO]}$.

CALCULUS.

121. Proposed by W. W. LANDIS, A. M., Professor of Mathematics and Astronomy, Dickinson College, Carlisle, Pa.

Solve the differential equation $\left[\frac{d}{dx}+b\right]^n y=\cos ax$.

Solution by LON C. WALKER, A. M., Petaluma High School, Petaluma, Cal., and LEWIS NEIKIRK, B. S., Boulder, Col.

$$\left[\frac{d}{dx}+b\right]^n y=\cos ax.$$

$\left[\frac{d}{dx}+b\right]^n$ has n roots each $=-b$.

$\therefore$ Comp. Factor $=e^{-bx}(c_1+c_2x+c_3x^2+c_4x^3+\ldots.c_nx^{n-1})$.

$$\frac{1}{\left[\frac{d}{dx}+b\right]^n}\cos ax=\frac{\left[\frac{d}{dx}-b\right]^n}{\left[\frac{d}{dx}-b^2\right]^n}\cos ax=\left\{\frac{b-\frac{d}{dx}}{a^2+b^2}\right\}^n\cos ax.$$

Let $n=1$. $\therefore$ $\frac{b-d/dx}{a^2+b^2}\cos ax=\frac{1}{a^2+b^2}(b\cos ax+a\sin ax)$

$$=\frac{1}{(a^2+b^2)^{\frac{1}{2}}}\left[\frac{b\cos ax+a\sin ax}{\sqrt{(a^2+b^2)}}\right]\ldots.(1).$$

Put $\theta=\cot^{-1}b/a$, then $\sin\theta=\frac{a}{\sqrt{(a^2+b^2)}}$, and $\cos\theta=\frac{b}{\sqrt{(a^2+b^2)}}$.

$\therefore$ (1) reduces to

$$\frac{b-d/dx}{a^2+b^2}\cos ax=(a^2+b^2)^{-\frac{1}{2}}(\cos\theta\cos ax+\sin\theta\sin ax)=(a^2+b^2)^{-\frac{1}{2}}\cos(ax-1\theta).$$

When $n=2$, $\dfrac{b-d/dx}{a^2+b^2}\left[\dfrac{1}{(a^2+b^2)^{\frac{1}{2}}}\cos(ax-\theta)\right]$

$$=\frac{1}{a^2+b^2}\left[\frac{b\cos(ax-\theta)+a}{\sqrt{(a^2+b^2)}}\sin(ax+\theta)\right]$$

$$=\frac{1}{a^2+b^2}\left[\cos\theta\cos(ax-\theta)+\sin\theta\sin(ax-\theta)\right]=(a^2+b^2)^{-\frac{2}{2}}\cos(ax-2\theta).$$

When $n=3$, $\therefore$ $\dfrac{b-d/dx}{a^2+b^2}\left[\dfrac{1}{(a^2+b^2)^{\frac{1}{2}}}\cos(ax-\theta)\right]$

$$=\frac{1}{a^2+b^2}\left[\frac{b\cos(ax-2\theta)a\sin(ax-2\theta)}{\sqrt{(a^2+b^2)}}\right]=(a^2+b^2)^{-\frac{3}{2}}\cos(ax-3\theta).$$

This method holds for $(n-1)$ terms.

Hence $y=e^{-bx}(c_1+c_2x+c_3x^2+c_4x^3+\ldots.c_nx^{n-1})$

$$+(a^2+b^2)^{-\frac{1}{2}n}\cos[ax-n\cot^{-1}(b/a)].$$

SECOND SOLUTION.

Put $D=\dfrac{d}{dx}$, then we have $\dfrac{1}{(D+b)^n}\cos ax=\dfrac{(D-b)^n}{(D^2-b^2)^n}\cos ax$

$$=\frac{(-1)^n(b-D)^n}{(a^2+b^2)^n}\cos ax.$$

The numerator$=(-1)^n\left[b^n\cos ax+nab^{-1}\sin ax)-\dfrac{n(n-1)}{2!}a^2b^{n-2}\cos ax\ldots\right]$

$$=(-1)^n\cos[ax-n\cot^{-1}(b/a)].$$

The value of y is the same as in I.

Also solved by *WILLIAM HOOVER*, *G. B. M. ZERR*, and the *PROPOSER*.

Professor Landis remarks that the exponent of a^2+b^2 is given $\frac{1}{2}n$ in Johnson's *Differential Equations* (problem 17, page 122) instead of the correct result $-\frac{1}{2}n$.

122. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

Solve the differential equation $(y-x)(1+x^2)\dfrac{dy}{dx}=n(1+y^2)$.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Let $x=\tan\theta$, $y=\tan\varphi$, then $\sin(\phi-\theta)d\varphi=nd\theta\ldots.(1)$.

$$(\tan\varphi-\tan\theta)\sqrt{(1+\tan^2\theta)}\frac{\sec^2\varphi d\varphi}{\sec^2\theta d\theta}=n(1+\tan^2\varphi)^{\frac{1}{2}}=n\sec^3\phi\ldots.(2).$$

$$(\tan\varphi-\tan\theta)d\phi=n\sec\phi\sec\theta d\theta\ldots.(3).$$

$$\frac{\sin\varphi\cos\theta-\cos\varphi\sin\theta}{\cos\varphi\cos\theta}d\varphi=\frac{nd\theta}{\cos\varphi\cos\theta}\ldots.(4).\quad n\frac{d\theta}{d\varphi}=\sin(\varphi-\theta)\ldots.(5).$$

Let $\omega=\phi-\theta$, then $\dfrac{d\omega}{1-(\sin\omega/n)}=d\varphi\ldots.(6).$

Whence $\cot\left[\dfrac{\pi}{4}-\dfrac{\varphi-\theta}{2}\right]=n(\varphi+c)\ \ldots.(7).$

$\cot[\frac{1}{4}\pi-\frac{1}{2}(\tan^{-1}y-\tan^{-1}x)]=n\tan^{-1}y+c_1$, or, let

$$y=\frac{x-z}{1+xz};\text{ then }\frac{z}{[1+z^2][z+n\sqrt{(1+z^2)}]}\frac{dz}{dx}-\frac{1}{1+x^2}=0,$$

in which the variables are separated.

II. Solution by **F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, O.**

Let $x=\tan\theta$, $y=\tan\varphi$, then $(\tan\varphi-\tan\theta)\sec\theta\dfrac{\sec^2\varphi}{\sec^2\theta}=n\sec^3\varphi.$

$\therefore\ \sec(\varphi-\theta)=n\dfrac{d\theta}{d\varphi}.$

Let $\omega=\varphi-\theta$, then $\dfrac{d\theta}{d\varphi}=1-\dfrac{d\omega}{d\varphi}.\quad\therefore\ \sin\omega=n-n\dfrac{d\omega}{d\varphi}.$

$$\therefore\frac{nd\omega}{n-\sin\omega}-d\varphi=0.$$

By the formulas given in the text books, if $n>1$, numerically,

$$\frac{2n}{\sqrt{(n^2-1)}}\tan^{-1}\frac{n\tan\frac{1}{2}\omega-1}{\sqrt{(n^2-1)}}=\varphi+c_1.$$

$$\therefore\frac{2n}{\sqrt{(n^2-1)}}\frac{n\tan\left(\frac{1}{2}\tan^{-1}\dfrac{y-x}{1+xy}\right)-1}{\sqrt{(n^2-1)}}=\tan^{-1}y+c_1\ ;\text{ if } n<1,\text{ numerically.}$$

$$\frac{1}{\sqrt{(1-n^2)}}\log\frac{n\tan\frac{1}{2}\omega-1-\sqrt{(1-n^2)}}{n\tan\frac{1}{2}\omega-1+\sqrt{(1-n^2)}}=\varphi+c_2.$$

$$\therefore\frac{1}{\sqrt{(1-n^2)}}\log\frac{n\tan\left(\frac{1}{2}\tan^{-1}\frac{y-x}{1+xy}\right)-1-\sqrt{(1-n^2)}}{n\tan\left(\frac{1}{2}\tan^{-1}\frac{y-x}{1+xy}\right)-1+\sqrt{(1-n^2)}}=\tan^{-1}y+c_2.$$

If $n=1$, $\frac{d\omega}{1-\sin\omega}-d\varphi=0.$

$$\therefore\frac{d\omega}{\sin^2\frac{1}{2}\omega+\cos^2\frac{1}{2}\omega-2\sin\frac{1}{2}\omega\cos\frac{1}{2}\omega}-d\varphi=0.$$

$$\therefore\frac{d\omega\sec^2\frac{1}{2}\omega}{(\tan\frac{1}{2}\omega-1)^2}-d\varphi=0.\quad\therefore\frac{2d(\tan\frac{1}{2}\omega-1)}{(\tan\frac{1}{2}\omega-1)^2}-d\varphi=0.$$

$$\therefore\frac{2}{1-\tan\frac{1}{2}\omega}=\varphi+c_2.\quad\therefore 1+\frac{\tan\frac{1}{2}\omega+1}{1-\tan\frac{1}{2}\omega}=\varphi+c_2.$$

$$\therefore\tan(\tfrac{1}{2}\omega+\tfrac{1}{4}\pi)=\varphi+c'^2.$$

$$\therefore\tan\left[\frac{\pi}{4}+\tfrac{1}{2}\tan^{-}\frac{y-x}{1+xy}\right]=\tan^{-1}y+c'_2\text{ ; if }n=-1,$$

$\frac{d\omega}{1+\sin\omega}-d\varphi=0$, and the result is $\tan\left[\frac{\pi}{4}-\frac{1}{2}\tan^{-1}\frac{y-x}{1+xy}\right]=\tan^{-1}y+c.$

Also solved by *L. C. WALKER*, *H. C. WHITAKER*, and *G. B. M. ZERR*.

123. Prize Problem. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

Find in finite terms, the value of $\int_0^{\frac{1}{2}\pi}\log\tan\phi d\phi.$

Solution by W. H. ECHOLS, Professor of Mathematics, University of Virginia, Charlottsville, Va.

Put $\tan z=x$. Then $\int_0^{\frac{1}{2}\pi}\log(\tan z)dz=\int_0^1\frac{\log x}{1+x^2}dx,$

$$=\int_0^1\log x(1-x^2+x^4-x^6+\ldots.)dx.$$

Differentiating, respecting a, $\int_0^1 x^a dx=\frac{1}{a+1}$.

$$\therefore\int_0^1 x^a\log x dx=-\frac{1}{(a+1)^2}.$$

$$\therefore \int_0^{\frac{1}{2}\pi} \log(\tan z)\,dz = -\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\ldots.\right) = -\frac{\pi^3}{32},$$

by a well known summation.

A correct result was also received from Lon. C. Walker. The integration in finite terms required that the entire work should be done in finite terms as is done, for example, in Byerly's *Integral Calculus*, page 98, when

$$\int_0^{\frac{1}{2}\pi} \log\sin x\,dx$$

is found, and not the final result stated in finite terms.

The summation of the above series may be found by using the relation,

$$\frac{B_{2n}}{(2n)!} = \frac{2^{2n+2}}{\pi^{2n+1}}\left[1-\frac{1}{3^{2n}}+\frac{1}{5^{2n}}-\frac{1}{7^{2n}}+\ \ldots\right]$$

When $n=1$, $B_2=1$, one of Euler's numbers. See B. O. Peirce's *Table of Integrals*, page 90. Ed.

124. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Show that the cardioides $r=a(1+\cos\theta)\ldots.(1)$, and $r=b(1-\cos\theta)\ \ldots(2)$, intersect at right angles.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.

The equation can be written $r=2a\cos^2\frac{1}{2}\theta$, $r=2b\sin^2\frac{1}{2}\theta$, or $r^{\frac{1}{2}}=(2a)^{\frac{1}{2}}\cos\frac{1}{2}\theta$, $r^{\frac{1}{2}}=(2b)^{\frac{1}{2}}\cos\frac{1}{2}(\pi-\theta)$.

$$\frac{dr}{rd\theta}=-\tan\tfrac{1}{2}\theta, \quad \frac{dr}{rd\theta}=+\tan\tfrac{1}{2}(\pi-\theta).$$

At the point of intersection r and θ are the same for both.

The angle made by the perpendicular from the origin on the tangent is in the first case $\frac{1}{2}\theta$, in the second $\frac{1}{2}(\pi-\theta)$.

But $\frac{1}{2}\theta+\frac{1}{2}(\pi-\theta)=\frac{1}{2}\pi$.

$\therefore$ These perpendiculars are perpendicular to one another.

$\therefore$ The tangents are perpendicular and the cardioids intersect at right angles.

Solved substantially as above by *JOHN F. TRAVIS*, Fellow and Assistant in Mathematics in Ohio State University, Columbus, O.; *E. L. SHERWOOD, A. M.*, Professor of Mathematics, Beaver College, Beaver, Pa., and *L. C. WALKER*.

MECHANICS.

125. Proposed by T. U. TAYLOR, C. E., Professor of Civil Engineering, University of Texas, Austin, Tex.

(1) If a parabola is described on the vertical face of a reservoir wall, axis vertical and in the surface, and $P(h, b)$ be any point on the curve, and B the foot of the perpendicular from P on the axis, find c. p. on area OBP.

(2) If A is point where horizontal through P cuts vertical axis (OY), find c. p. on area OAP.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

From statement of problem "axis vertical" should evidently read "axis horizontal."

Let $y^2=2px$ be the parabola. But $b^2=2ph$.

$\therefore\ y^2=b^2x/h$. Let p=pressure, m=moment, w=unit weight of water.

(1) $dp=wy(h-x)dy$, $dm=wy^2(h-x)dy$.

$$\therefore\ \bar{y}=\frac{\int_{-b}^{0}y^2(h-x)dy}{\int_{-b}^{0}y(h-x)dy}=\frac{\int_{-b}^{0}y^2(h-hy^2/b^2)dy}{\int_{-b}^{0}y(h-hy^2/b^2)dy}=-\tfrac{8}{15}b.$$

$$\bar{x}=\frac{\int_0^h y(h-x)^2dx}{\int_0^h y(h-x)dx}=\frac{\int_0^h \sqrt{x}\,(h-x)^2dx}{\int_0^h \sqrt{x}\,(h-x)dx}=\tfrac{4}{7}h.$$

(2) The depth of an elementary strip length x is y.

$\therefore\ dp=wxydy$, $dm=wxy^2dy$.

$$\therefore\ \bar{y}=\frac{\int_{-b}^{0}xy^2dy}{\int_{-b}^{0}xydy}=\frac{\int_{-b}^{0}y^4dy}{\int_{-b}^{0}y^3dy}=-\tfrac{4}{5}b.$$

$$\bar{x}=\frac{\int_0^h x^2(b-y)dx}{\int_0^h x(b-y)dx}=\frac{\int_0^h x^2\left(b-\frac{b\sqrt{x}}{\sqrt{h}}\right)dx}{\int_0^h x\left(b-\frac{b\sqrt{x}}{\sqrt{h}}\right)dx}=\frac{\int_0^h x^2(\sqrt{h}-\sqrt{x})dx}{\int_0^h x(\sqrt{h}-\sqrt{x})dx}=\tfrac{5}{21}h.$$

126. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

AB is the horizontal base of a smooth cycloidal tube, vertex downward. A sphere is placed in the tube at A, and when it reaches the vertex another sphere of different mass is placed in the tube at B. When and where do they meet, and find their velocity immediately after collision, the spheres being partially elastic?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let a be the radius of the generating circle, e the coefficient of restitution. m=mass of first sphere, m_1=mass of second sphere. It has been demonstrated that the first sphere will reach the vertex in the time, $t=\pi\sqrt{(a/g)}$, with a velocity, $v=2\sqrt{(ag)}$.

To descend half the vertical height it requires $\frac{1}{2}\pi\sqrt{(a/g)}$ seconds.

$\therefore$ The spheres meet at a vertical distance equal to the radius of the generating circle above the vertex, $\frac{1}{2}\pi\sqrt{(a/g)}$ seconds after the second sphere is placed in the tube.

At the moment of impact both spheres have a velocity$=\sqrt{(2ag)}$. Let v_1, v' be the velocity of the first and second spheres after impact. Then for direct impact $v_1-v'=e[\sqrt{(2ag)}-\sqrt{(2ag)}]=0$.

$\therefore\ v_1=v'$, also $\sqrt{(2ag)}(m+m_1)=mv_1+m_1v'$.

$\therefore\ v_1=v'=\sqrt{(2ag)}$.

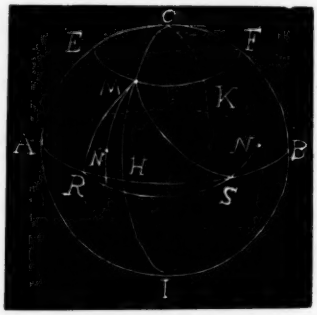
AVERAGE AND PROBABILITY.

107. Proposed by LON C. WALKER. A. M., Professor of Mathematics, Petaluma High School. Petaluma. Cal.

Two points are taken at random in the curved surface of a hemisphere. Show (1) that the average length of the straight therein is $32r/9\pi$; and (2) that the average length of the arc of a great circle, which joins them, is $4r/\pi$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let m, n be two random points on the same side of the great circle $AHBK$. Through M draw the great circle $CH1K$ perpendicular to $AHBK$, and the small circle EMF parallel to $AHBK$. From M as a pole with a spherical radius equal to MN describe the arc RSN intersecting $AHBK$ in R and S.



Let arc $MN=r\theta$, $HM=r\varphi$, $RH=HS=r\mu$, $\angle RMH=\psi$.

Then $\sin\mu=\sin\theta\sin\psi$....(1).

$\cos\psi=\tan\varphi\cot\theta$....(2).

$\cos\theta=\cos\varphi\cos\mu$....(3).

Length of straight$=2r\sin\frac{1}{2}\theta$.

An element of surface at M is $2\pi r\cos\varphi r d\varphi$.

If θ is less than $\frac{1}{2}\pi$, for each position of M, N may be taken anywhere on the supplement of the arc RNS, if φ varies from 0 to θ. If θ is greater than $\frac{1}{2}\pi$, while φ varies from 0 to $\pi-\theta$, N can be taken on the same arc. The length of this arc is $2(\pi-\psi)r\sin\theta$. For φ from 0 to $\frac{1}{2}\pi$ and θ from 0 to φ, N can be taken anywhere on the circumference $2\pi r\sin\theta$. Then

$$(1)\quad J=\frac{1}{4\pi^2r^4}\int_0^{\frac{1}{2}\pi}\int_0^{\theta} 2r\sin\tfrac{1}{2}\theta.2(\pi-\psi)r\sin\theta r d\theta.2\pi r\cos\varphi r d\varphi$$

$$+\frac{1}{4\pi^2r^4}\int_{\frac{1}{2}\pi}^{\pi}\int_0^{\pi-\theta} 2r\sin\tfrac{1}{2}\theta.2(\pi-\psi)r\sin\theta r d\theta.2\pi r\cos\varphi r d\varphi$$

$$+\frac{1}{4\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_0^{\phi} 2r\sin\tfrac{1}{2}\theta.2\pi r\cos\varphi r d\varphi.2\pi r\sin\theta r d\theta.$$

$$\therefore \Delta = 2r\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\sin\tfrac{1}{2}\theta\sin\theta\cos\varphi d\theta d\varphi + 2r\int_{\frac{1}{2}\pi}^{\pi}\int_{\frac{1}{2}\pi}^{\pi-\theta}\sin\tfrac{1}{2}\theta\sin\theta\cos\varphi d\theta d\varphi$$

$$+2r\int_0^{\frac{1}{2}\pi}\int_0^{\phi}\cos\varphi\sin\tfrac{1}{2}\theta\sin\theta d\varphi d\theta - \frac{2r}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\sin\tfrac{1}{2}\theta\sin\theta d\theta.\psi\cos\varphi d\varphi$$

$$-\frac{2r}{\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_0^{\pi-\theta}\sin\tfrac{1}{2}\theta\sin\theta d\theta.\psi\cos\varphi d\varphi.$$

Now $\int\psi\cos\varphi d\varphi = \psi\sin\varphi - \int\sin\varphi d\psi.$

From (1), $\sin\theta\cos\psi d\psi = \cos\mu d\mu$ ....(4).
From (2) and (4), $\cos\theta\sin\varphi d\psi = \cos\varphi\cos\mu d\mu$ ....(5).
From (3) and (5), $\sin\varphi d\psi = d\mu$.

$$\therefore \int\psi\cos\varphi d\varphi = \psi\sin\varphi - \mu.$$

When $\varphi=0$, $\psi=\frac{1}{2}\pi$, $\mu=\theta$; when $\varphi=\theta$, $\psi=0$, $\mu=0$; when $\varphi=\pi-\theta$, $\psi=\pi$, $\mu=\pi$.

$$\therefore \Delta = 2r\int_0^{\frac{1}{2}\pi}\sin\tfrac{1}{2}\theta\sin^2\theta d\theta + 2r\int_{\frac{1}{2}\pi}^{\pi}\sin\tfrac{1}{2}\theta\sin\theta d\theta$$

$$+\tfrac{8}{3}r\int_0^{\frac{1}{2}\pi}\sin^3\tfrac{1}{2}\varphi\cos\varphi d\varphi - \frac{2r}{\pi}\int_0^{\pi}\theta\sin\tfrac{1}{2}\theta\sin\theta d\theta.$$

$$\therefore \Delta = 32/9\pi.$$

(2) In this case write θ for $2\sin\frac{1}{2}\theta$.

$$\therefore \Delta = r\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\theta\sin\theta\cos\varphi d\theta d\varphi + r\int_{\frac{1}{2}\pi}^{\pi}\int_0^{\pi-\theta}\theta\sin\theta\cos\varphi d\theta d\varphi$$

$$+r\int_0^{\frac{1}{2}\pi}\int_0^{\phi}\cos\varphi d\varphi.\theta\sin\theta d\theta - \frac{r}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\theta\sin\theta d\theta.\psi\cos\varphi d\varphi$$

$$-\frac{r}{\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_0^{\pi-\theta}\theta\sin\theta d\theta.\psi\cos\varphi d\varphi.$$

$$\therefore \Delta = r\int_0^{\frac{1}{2}\pi}\theta\sin^2\theta d\theta + r\int_{\frac{1}{2}\pi}^{\pi}\theta\sin\theta d + r\int_0^{\frac{1}{2}\pi}(\sin\varphi - \varphi\cos\varphi)\cos\varphi d\varphi$$

$$-\frac{r}{\pi}\int_0^{\pi}\theta^2\sin\theta d\theta = 4r/\pi.$$

The first result is the same as the average distance of a point in a circle from a point in its circumference. The second result is the same as the average distance between two points on the circumference of a circle.

Also solved by **F. P. MATZ**. Professor Matz in his solution did not go through the details of integration as Professor Zerr has done.

II. Solution by **L. C. WALKER**, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Let P, Q be the random points, O the center of the hemisphere, and A the vertex. Put $OP=r$, $\angle POQ=\theta$, $\angle AOP=\varphi$, M_1=average length of the straight line, and M_2=average length of the arc of a circle, which joins the points.

Now while θ is constant, and $<\frac{1}{2}\pi$, and $\phi<\frac{1}{2}\pi-\theta$, for each position of P, Q may be taken anywhere in the circumference of a small circle whose pole is P, and radius $r\sin\theta$.

But when $\theta<\frac{1}{2}\pi$, and $\varphi>\frac{1}{2}\pi-\theta$, or when $\theta>\frac{1}{2}\pi$, and $\phi>\theta-\frac{1}{2}\pi$, for each position of P, Q may be taken anywhere in the arc of a small circle whose pole is P, and length $2r\sin\theta[\pi-\cos^{-1}(\cot\theta\cot\phi)]$.

Hence, if θ is of given value, and $<\frac{1}{2}\pi$, the number of ways the two points can be taken is

$$\int_0^{\frac{1}{2}\pi-\theta} 2\pi r\sin\theta.2\pi r\sin\phi.rd\phi+\int_{\frac{1}{2}\pi-\theta}^{\frac{1}{2}\pi} 2r\sin\theta[\pi-\cos^{-1}(\cot\theta\cot\phi)].2\pi r\sin\phi.rd\phi$$

$$=4\pi r^3(\pi-\theta)\sin\theta.$$

If $\theta>\frac{1}{2}\pi$, the number of ways the two points can be taken is

$$\int_{\theta-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 2r\sin\theta[\pi-\cos^{-1}(\cot\theta\cot\phi)].2\pi r\sin\phi.rd\phi=4\pi r^3(\pi-\theta)\sin\theta.$$

Hence, since the whole number of ways the two points can be taken is $4\pi^2r^4$, we have

$$M_1=\frac{1}{4\pi^2r^4}\int_0^{\pi} 4\pi r^3(\pi-\theta)\sin\theta.2r\sin\tfrac{1}{2}\theta.rd\theta=\frac{32r}{9\pi}, \text{ and}$$

$$M_2=\frac{1}{4\pi^2r^4}\int_0^{\pi} 4\pi r^3(\pi-\theta)\sin\theta.r\theta.rd\theta=\frac{4r}{\pi}.$$

MISCELLANEOUS.

101. Proposed by **G. B. M. ZERR**, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

A wire is laid along the surface of a right cone semi-vertical angle β so that it cuts the generators everywhere at a constant angle θ. Find the radius of curvature and radius of torsion.

Solution by the PROPOSER.

Let s=length of wire from origin to any point, ρ=radius of curvature, $1/\sigma$=radius of torsion. From problem 85, Calculus, No. 4, Vol. VI., we get $x=r\cos\varphi$, $y=r\sin\varphi$, $z=s\cos\beta\cos\theta$, $\varphi=\frac{\tan\theta}{\sin\beta}\log s$, $r=s\sin\beta\cos\theta$, $dx=\cos\varphi dr-r\sin\varphi d\varphi$,

$dy=\sin\varphi dr+r\cos\varphi d\varphi$, $dz=\cos\beta\cos\theta ds$, $d\varphi=\frac{\tan\theta}{\sin\beta}\cdot\frac{ds}{s}$, $dr=\sin\beta\cos\theta ds$.

$\therefore\ dx/ds=\sin\beta\cos\theta\cos\varphi-\sin\theta\sin\varphi$, $dy/ds=\sin\beta\cos\theta\sin\varphi+\sin\theta\cos\varphi$,

$dz/ds=\cos\beta\cos\theta$, $d^2x/ds^2=-\frac{\sin\theta}{s}\left(\sin\varphi+\frac{\tan\theta\cos\varphi}{\sin\beta}\right)$,

$d^2y/ds^2=\frac{\sin\theta}{s}\left(\cos\varphi-\frac{\tan\theta\sin\varphi}{\sin\beta}\right)$, $d^2z/ds^2=d^3z/ds^3=0$,

$d^3x/ds^3=-\frac{\sin\theta\tan\theta}{s^2\sin\beta}\left(\cos\varphi-\frac{\tan\theta\sin\varphi}{\sin\beta}\right)$, $d^3y/ds^3=-\frac{\sin\theta\tan\theta}{s^2\sin\beta}\left(\sin\varphi+\frac{\tan\theta\cos\varphi}{\sin\beta}\right)$

$$1/\rho^2=(d^2x/ds^2)^2+(d^2y/ds^2)^2+(d^2z/ds^2)^2=\frac{\sin^2\theta}{s^2}\left(1+\frac{\tan^2\theta}{\sin^2\beta}\right).$$

$$\frac{1}{\rho^2\sigma}=\begin{vmatrix}\frac{dx}{ds}, & \frac{dy}{ds}, & \frac{dz}{ds}\\ \frac{d^2x}{ds^2}, & \frac{d^2y}{ds^2}, & \frac{d^2z}{ds^2}\\ \frac{d^3x}{ds^3}, & \frac{d^3y}{ds^3}, & \frac{d^3z}{ds^3}\end{vmatrix}=\frac{dz}{ds}\left(\frac{d^2x}{ds^2}\cdot\frac{d^3y}{ds^3}-\frac{d^3x}{ds^3}\cdot\frac{d^2y}{ds^2}\right)$$

$$=\frac{\sin^2\theta\tan\theta}{s^3\sin\beta}\left(1+\frac{\tan^2\theta}{\sin^2\beta}\right)\cos\beta\cos\theta.$$

$$\therefore\ 1/\sigma=\frac{\tan\theta\cos\beta\cos\theta}{s\sin\beta}=\frac{\sin\theta\cos\beta}{s\sin\beta}.$$

Also solved by *WILLIAM HOOVER*.

102. Proposed by CHARLES C. CROSS, Whaleyville, Va.

Required the least multiple of 17 which when divided by 2, 3, 4, . . . 16 leaves in each case 1 as a remainder.

I. Solution by MARVIN E. SMITHEY, A. M., Instructor in Mathematics, Randolph-Macon Academy, Bedford City, Va.

$x(2^4.3^2.5.7.11.13)+1$, x being any positive integer, represents all numbers satisfying the given condition. In order to determine what value of x will make this number the least multiple of 17, the following equation must be solved in least positive integers.

$$720720x+1=17y, \quad y=42395x+(5x+1)/17.$$

Let $(5x+1)/17=m$, an integer.

$\therefore\ x=(17m-1)/5.$

Make $m=3$, then $x=10$. The required number is $(10)720720+1=7207210$.

Solved in a similar manner of *L. M. COFFIN*, University of Maine, Orono, Me.; *J. CHAS. RATHBUN*, Student State University of Washington, Seattle, Wash.; and *G. B. M. ZERR*.

II. Solution by **M. A. GRUBER, A. M., War Department, Washington, D. C.**

The problem may be generally stated: $an \div 2, 3, 4, 5, \ldots (a-1)$, respectively, gives a remainder of 1; where a=any prime odd number.

Let M=least common multiple of 2, 3, 4, 5. .... $(a-1)$.

Then $an=Mm+1$, and $n=\dfrac{Mm+1}{a}=Qm+\dfrac{dm+1}{a}$.

To render $\dfrac{dm+1}{a}$ integral, we put $\dfrac{dm+1}{a}=d(p-1)+c$, where p=any integer, and $c=n$ in the *first multiple* of a that renders m integral in the equation $dm+1=ac$.

When $m=a(p-1)+\dfrac{ac-1}{d}$; or the least value of m is $\dfrac{ac-1}{d}$, and every subsequent value of m is a greater than the value just preceding.

$\therefore\ an=M[a(p-1)+\dfrac{ac-1}{d}]+1$; the least value being obtained when $p=1$.

Also solved by *JOHN G. KELLAR, HARRY S. VANDIVER, L. C. WALKER, H. C. WHITAKER*, and the *PROPOSER*.

103. Proposed by **ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.**

Solve, $\log\sin x=\sin\log x$.

Solution by **H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.**

1. Suppose that imaginary values are neglected; then $\sin x$ must be positive and therefore x is $2n\pi$ and $(2n+1)\pi$, n being a whole number either positive, negative, or zero.

2 Since $\sin x$ is less than 1, $\log\sin x$ is negative, therefore $\sin\log x$ is also negative and hence $\log x$ is between $(2m+1)\pi$ and $2(m+1)\pi$; taking $m=0$, the limits of x for this condition are e^{π} and $e^{2\pi}$ or 23.146 and 535.5. By the first condition, numbers between 23 and 535.5 when divided by 6.2832 must leave a remainder less than 3.1416; the least positive value of x satisfying both conditions is then found to be between 25.1328 and 26.704. By trial this least positive value of x is found to be 26.208. Another value exists between 26.704 and 28.274 (about 27.4 by trial). Another value between 31.416 and 34.557, and so on, the successive limits for each new value of x being found by adding 6.2832 to the limits already found until 535.5 is reached. After that no real values of x are obtainable until 12392 $(=e^{3\pi})$ is reached; then two more will occur for each increase of 6.2832 until 286751 $(=e^{4\pi})$ is reached—and so on.

In a similar manner, negative values of x may be considered.

Also solved by *G. B. M. ZERR*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

152. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

An operator on 'Change gains 5% on his *daily capital* every *odd* day of a business-week, and loses 5% of the same capital every *even* day of same week. What per cent. of his *original capital* will he have gained, or lost, at the end of a business-week?

153. Proposed by J. C. CORBIN, Pine Bluff, Ark.

Find some two-figure numbers, such that if they be squared, then the figures interchanged and the resulting numbers squared, the resulting products will consist of the same digits in reversed order.

ALGEBRA.

153. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Eliminate x, y, z from the equations,

$$x^2+yz=a,$$
$$y^2+xz=b,$$
$$z^2+xy=c,$$
$$x+y+z=0.$$

154. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Show that the equation, $x^4+qx^2+s=0$ ...(1), can not have three *equal* roots.

155. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

If the roots of the cubic $x^3+3px^2+3qx+r=0$ be in harmonical progression, $2q^3=r(3pq-r)$.

156. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

$(z+x)a-(z-x)b=2yz$.

CALCULUS.

145. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Find the surface bounding the volume required in problem 102.

146. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

$$I=\int_0^{\frac{1}{2}\pi} \sqrt{[(p+\tfrac{1}{4}q^2)-(\tfrac{1}{2}q+\cos\theta)^2]}\,d\theta=\text{what ?}$$

147. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Find volume common to the two solids

$$(x/a)^{\frac{3}{2}} + (y/b)^{\frac{3}{2}} = (z/c)^{\frac{3}{2}}, \quad (y/b)^{\frac{3}{2}} + (z/c)^{\frac{3}{2}} = (x/a)^{\frac{3}{2}}.$$

MECHANICS.

136. Proposed by F. T. WRIGHT, Ph. B., Schenectady, N. Y.

In an air brake test a train moving at 22 miles an hour on a down grade of one per cent. was stopped in 91 feet. There was 94 per cent. of the train braked. Taking the fractional resistance as 8 pounds per ton, find the net brake resistance per ton.

137. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

A uniform inextensible string rests against the inner side of a smooth elliptic wire semi-axes a and b, and is repelled from the foci and the center by the following forces: μ/rd and $\nu/r'd$ emanating from the foci, and $\pi c/d$ from the center, the distances of any point on the string from the foci being r and r', respectively, its distance from the center being c, and the semi-conjugate diameter corresponding to the point being d. Find the pressure on the wire at any point.

DIOPHANTINE ANALYSIS.

99. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

Find a general expression for the radius of the sphere which, dropped in (or partly in) a right cone full of water, will displace the most water; the radius of the sphere, and the width, height and slant height of the cone to be rational integral numbers.

100. Proposed by LON C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

(*a*) Find the least three integral numbers such, that if to the square of each the product of the other two be added, the three sums shall be all squares.

(*b*) Find the two least integral numbers such, that not only each of them, but also their sum and their difference, when increased by *unity*, shall be all square numbers.

AVERAGE AND PROBABILITY.

121. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Find the average area of the pentagon formed by joining five random points on the surface of a given circle.

122. Proposed by F. M. PRIEST, St. Louis, Mo.

Suppose each of the nine digits to be placed in a wheel, and five of them drawn at random therefrom, and written down in the order drawn. What is the probability the number thus expressed will be greater than 50,000?

MISCELLANEOUS.

121. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

How can we determine the elements of a cyclone from observations made at three different points?

122. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

How should a Division of Space be made in order that the *Partional Area* may be a Minimum?

123. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

If a *curve* of the third degree can not be made to pass through more than six arbitrarily chosen points, why can a *surface* of the third degree be made to pass through nineteen such points?

NOTES.

THE MONTHLY is mailed on the 28th of each month and should reach its subscribers soon after that time.

The notice of the new Postal ruling which appeared in our last issue was somewhat premature. We had received a number of notices to that effect, but none of them were official.

This number of THE MONTHLY is sent to all of our old subscribers who have not yet renewed their subscriptions as well as to those who have renewed. Any one who does not wish to continue his subscription will please return this number with his name plainly written on the wrapper.

Owing to the demands on his time in the preparation of other mathematical text-books, of which the Colaw and Ellwood Arithmetics form a part, Professor Colaw's connection with THE MONTHLY will be temporarily severed. In view of this fact all contributions should be sent to B. F. Finkel until further notice is given.

We desire to thank all of our subscribers and contributors for the many kind words that have come to us to encourage us in our work. We also desire to thank all of our subscribers who so promptly responded to our notices for renewals. To renew promptly is one of the easy ways in which every one can very materially assist us in carrying on the good work of THE MONTHLY, for it should be remembered that our financial problem is one which gives us much anxiety.

Dr. William Anthony Granville, Instructor in Mathematics in Sheffield Scientific School of Yale University, has designed a plotting paper, The Polar Coördinate Plotting Paper. Every teacher of Analytical Geometry must often have felt the need of such paper. But not only is this paper serviceable to the students of Analytical Geometry and Calculus, but also in the solution of problems in Vector Analysis, Mechanics, Astronomy, and Engineering. The price of the paper is about the same as for ordinary rectangular plotting paper. Persons desiring to use this paper should write to Dr. Granville.

BOOKS.

An Elementary Treatise on the Calculus, with illustrations from Geometry, Mechanics, and Physics. By George A. Gibson, M. A., F. R. S. E., Professor of Mathematics in the Glasgow and West of Scotland Technical College. Cloth, 12mo., xix+459 pages. Price, $1.90, net. New York: The Macmillan Co.

In writing this work, the aim of the author seems to have been to prepare the student for immediately applying the principles and processes of the Calculus in any department of his studies in which the Calculus is used. With this end in view he has illustrated the applications of the Calculus by drawing on Geometry, Mechanics, and Physics. We heartily approve of this method of treating the subject. The Calculus is being studied by a larger number of students to-day than ever before, and by its making use of illustrations in such apparently unrelated subjects as Physics and Chemistry, greater interest is aroused. This book emphasizes the fact that even in such a subject as Chemistry a sound knowledge of the Calculus is of especial importance, since it is the properties of functions of more than one variable that are predominant in chemical investigations. The book closes with a short chapter on Ordinary Diflerential Equations, designed to illustrate the types of equations most frequently met with in dynamics, physics, and mechanical and electrical engineering. B. F. F.

The Groups of Steiner in Problems of Contact, by Dr. Leonard E. Dickson. Reprinted from the Transactions of the American Mathematical Society.

Representation of Linear Groups as Transitive Substitution Groups. By Dr. Leonard E. Dickson. Reprinted from the American Journal of Mathematics, Vol. XXIII, No. 4.

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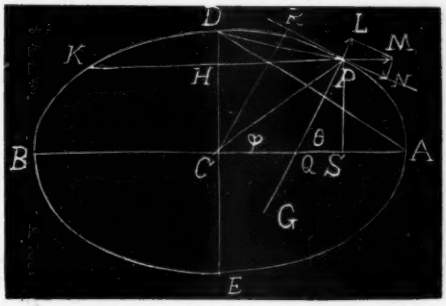
GRAVITY, TRUE AND APPARENT.

By G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

Let P be any point on the earth's surface, $CA=CB=a$, the earth's equatorial radius, $CD=CE=b$, the earth's polar radius, (x, y) the coördinates of P, $CP=r$, $\angle PQA=\theta$=latitude of P, $\angle PCA=\varphi$, mass of particle at P=unity, f=centrifugal force of particle at P, e=earth's eccentricity, g'=true gravity in any latitude, g=apparant gravity in any latitude, G'=true gravity at the equator, G=apparant gravity at the equator, l'=length of second's pendulum for true gravity in any latitude, l=length of second's pendulum for apparent gravity in any latitude, L'=length of second's pendulum for true gravity at the equator, L=length of second's pendulum for apparent gravity at the equator, T=number of seconds in a siderial day.



Then $HP=r\cos\varphi=x$, $CH=r\sin\varphi=y$, $HP=CQ+QS$, $CQ=e^2x=e^2r\cos\varphi$.

$$PQ=\frac{b}{a}\sqrt{(a^2-e^2x^2)}=\frac{1}{a}\sqrt{(b^4/a^2+e^2y^2)}=\frac{b}{a}\sqrt{(a^2-e^2r^2\cos^2\varphi)}$$

$$=\sqrt{(b^4/a^2+e^2r^2\sin^2\varphi)},\ QS=PQ\cos\theta.$$

$$\therefore\ r\cos\varphi=e^2r\cos\varphi+\frac{b\cos\theta}{a}\sqrt{(a^2-e^2r^2\cos^2\varphi.)}$$

$$\therefore\ r\cos\varphi=\frac{a^2\cos\theta}{\sqrt{(b^2+a^2e^2\cos^2\theta)}}=\frac{a\cos\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

$$PS=CH=r\sin\varphi=PQ\sin\theta=\sin\theta\sqrt{(b^4/a^2+e^2r^2\sin^2\varphi)}.$$

$$\therefore\ r\sin\varphi=\frac{b^2\sin\theta}{a\sqrt{(1-e^2\sin^2\theta)}}=\frac{b\sqrt{(1-e^2)}\sin\theta}{\sqrt{(1-e^2\sin^2\theta)}}.\quad f=v^2/HP=\frac{4\pi^2HP}{T^2}.$$

The component of f along PQ is $PL=f\cos\theta$.

$$\therefore\ f\cos\theta=\frac{4\pi^2HP\cos\theta}{T^2}=\frac{4a\pi^2\cos^2\theta}{T^2\sqrt{(1-e^2\sin^2\theta)}}.$$

According to the spheroid of Listing of Göttingen, 1878, a=6377377 meters=20923536 feet, b=6355270 meters=20851005 feet.

$$\therefore\ e^2=\frac{a^2-b^2}{a^2}=.00692.\quad \text{But } T=86164 \text{ seconds.}$$

$$\therefore\ f\cos\theta=\frac{.1112553\cos^2\theta}{\sqrt{(1-e^2\sin^2\theta)}}=.1112553[1-(1-\tfrac{1}{2}e^2)\sin^2\theta] \text{ nearly,}$$

$$=.1112553(1-.99654\sin^2\theta) \text{ feet per second.}$$

According to Newcomb and Holden the true gravity is given by $g'=G'(1+.00173\sin^2\theta)$ for any latitude.

Now $g'-f\cos\theta=g$, also $G'=32.2015235$.

$$\therefore\ g=32.2015235(1+.00173\sin^2\theta)-.1112553(1-.99654\sin^2\theta)$$

$$=32.0902682(1+.00519\sin^2\theta)=G(1+.00519\sin^2\theta).$$

$.00173=\frac{1}{4}e^2$, $.00519=\frac{3}{4}e^2$.

$$\therefore\ g'=G'(1+\tfrac{1}{4}e^2\sin^2\theta),\ g=G(1+\tfrac{3}{4}e^2\sin^2\theta).$$

In all previous writings these formulae are given in the form $g=g_0(1+c\sin^2\theta)$ where c is a constant. Whether this constant is a function of any dimension of the spheroid or whether it has any connection with the spheroid is not stated. I here express both constants in terms of the eccentricity. Dr. Mendenhall, the greatest American authority on gravity, has told me that these constants were never before so expressed.